PLEASE NOTE ANSWERS TO SOME PROBLEMS ARE NOT UNIQUE!

WNE Linear Algebra Final Exam
Series A

9 February 2017

Please use separate sheets for different problems. Please provide the following data on each sheet

- **name, surname and your student number,**
- **number of your group,**
- **number of the corresponding problem.**

Problem 1.
Let $v_1 = (1,1,1,1), v_2 = (2,1,2,3), v_3 = (1,0,1,t)$ be vectors in $\mathbb{R}^4$.

a) for which $t \in \mathbb{R}$ vectors $v_1, v_2, v_3 \in \mathbb{R}^4$ are linearly independent?
b) find a system of linear equations which set of solutions is equal to $\mathrm{lin}(v_1, v_2, v_3)$ for $t = 3$.

Solution.
Put vectors horizontally in a matrix and perform elementary row operations to get an echelon form.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 3 \\ 1 & 0 & 1 & t \end{bmatrix} \xrightarrow[r_3-r_1]{r_2-2r_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & t-1 \end{bmatrix} \xrightarrow{\substack{r_1+r_2 \\ r_3-r_2 \\ (-1)\cdot r_2}} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & t-2 \end{bmatrix}$$

a) elementary operations do not change linear dependence. If $t \neq 2$ then

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & t-2 \end{bmatrix} \xrightarrow{\substack{r_3/(t-2) \\ r_2+r_3 \\ r_1-2r_3}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

vectors $(1,0,1,0), (0,1,0,0), (0,0,0,1)$ are linearly independent, so are v_1, v_2, v_3. For $t = 2$ we get the zero vector so vectors v_1, v_2, v_3 are linearly dependent.
Answer: $t \neq 2$

b) for $t = 3$ any vector in $\mathrm{lin}(v_1, v_2, v_3)$ is equal to $x_1(1,0,1,0) + x_2(0,1,0,0) + x_4(0,0,0,1) = (x_1, x_2, x_1, x_4)$ for some $x_1, x_2, x_4 \in \mathbb{R}$. This is a general solution of the following system consisting of a single linear equation

$$x_1 - x_3 = 0$$

Answer:

$$x_1 - x_3 = 0$$

Problem 2.
Let $W \subset \mathbb{R}^5$ be a subspace given by the homogeneous system of linear equations

$$\begin{cases} x_1 + x_2 + 2x_3 - x_4 + 2x_5 = 0 \\ x_1 + x_2 + 3x_3 + x_4 + 3x_5 = 0 \\ 2x_1 + 3x_2 + 5x_3 - 3x_4 + 3x_5 = 0 \end{cases}$$

a) find a basis $\mathcal{A}$ of the subspace W and the dimension of W,
b) complete the basis $\mathcal{A}$ to a basis $\mathcal{B}$ of $\mathbb{R}^5$ and find coordinates of $w = (1,0,0,0,0) \in \mathbb{R}^5$ relative to $\mathcal{B}$.

Solution.
Solve the system of linear equations by bringing the matrix of coefficients to a reduced echelon form

$$\begin{bmatrix} 1 & 1 & 2 & -1 & 2 \\ 1 & 1 & 3 & 1 & 3 \\ 2 & 3 & 5 & -3 & 3 \end{bmatrix} \xrightarrow[r_3-2r_1]{r_2-r_1} \begin{bmatrix} 1 & 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & -1 & -1 \end{bmatrix} \xrightarrow{r_1-r_3} \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & -1 & -1 \end{bmatrix} \xrightarrow[r_3-r_2]{r_1-r_2}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & -3 & -2 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 0 & 0 & -2 & 2 \\ 0 & 1 & 0 & -3 & -2 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

The general solution is

$$\begin{cases} x_1 = 2x_4 - 2x_5 \\ x_2 = 3x_4 + 2x_5 \\ x_3 = -2x_4 - x_5 \end{cases}$$

that is $(2x_4-2x_5, 3x_4+2x_5, -2x_4-x_5, x_4, x_5) = x_4(2,3,-2,1,0)+x_5(-2,2,-1,0,1)$, $x_4, x_5 \in \mathbb{R}$.

a) **Answer:** The basis of W is $\mathcal{A} = ((2,3,-2,1,0),(-2,2,-1,0,1))$ and $\dim W = 2$.
b) observe that the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 3 & -2 & 1 & 0 \\ -2 & 2 & -1 & 0 & 1 \end{bmatrix}$$

can be brought by the elementary row operations to the identity matrix (alternatively, its determinant is non–zero), therefore rows of it give a basis of $\mathbb{R}^5$. It is easy to see that

$$w = 1(1,0,0,0,0)+0(0,1,0,0,0)+0(0,0,1,0,0)+0(2,3,-2,1,0)+0(-2,2,-1,0,1)$$

Answer: The basis is $\mathcal{B} = ((1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(2,3,-2,1,0),(-2,2,-1,0,1))$. The coordinates of w relative to $\mathcal{B}$ are $1,0,0,0,0$.

Problem 3.
Let $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation given by the formula

$$\varphi((x_1,x_2,x_3)) = (-4x_1 + x_2 + 2x_3, tx_2, -x_1 + x_2 - x_3).$$

a) for $t = -3$ find matrix $C \in M(3 \times 3;\mathbb{R})$ such that matrix $C^{-1}M(\varphi)_{st}^{st}C$ is diagonal,

b) find all $t \in \mathbb{R}$ for which there exist a basis $\mathcal{A}$ of $\mathbb{R}^3$ such that $M(\varphi)_{\mathcal{A}}^{\mathcal{A}} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & q \end{bmatrix}$, where $p, q \in \mathbb{R}$.

Solution.

a)

$$M(\varphi)_{st}^{st} = \begin{bmatrix} -4 & 1 & 2 \\ 0 & -3 & 0 \\ -1 & 1 & -1 \end{bmatrix}, \quad w_\varphi(\lambda) = \det \begin{bmatrix} -4-\lambda & 1 & 2 \\ 0 & -3-\lambda & 0 \\ -1 & 1 & -1-\lambda \end{bmatrix}.$$

$$w_\varphi(\lambda) = (-1)^{2+2}(-3-\lambda)\det \begin{bmatrix} -4-\lambda & 2 \\ -1 & -1-\lambda \end{bmatrix} = -(\lambda+3)^2(\lambda+2).$$

The eigenvalues are $\lambda = -2$ and $\lambda = -3$. Compute eigenspaces

$$V_{(-2)}: \begin{bmatrix} -2 & 1 & 2 \\ 0 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff x_2 = 0,\ x_1 = x_3,\ x_3 \in \mathbb{R}$$

$V_{(-2)} = \{(x_3, 0, x_3) \in \mathbb{R}^3 \mid x_3 \in \mathbb{R}\} = \operatorname{lin}((1,0,1))$

$$V_{(-3)}: \begin{bmatrix} -1 & 1 & 2 \\ 0 & 0 & 0 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff x_1 = x_2 + 2x_3,\ x_2, x_3 \in \mathbb{R}$$

$V_{(-3)} = \{(x_2 + 2x_3, x_2, x_3) \in \mathbb{R}^3 \mid x_2, x_3 \in \mathbb{R}\} = \operatorname{lin}((1,1,0),(2,0,1))$
There exist basis $\mathcal{A} = ((1,0,1),(1,1,0),(2,0,1))$ or $\mathbb{R}^3$ consisting of eigenvectors of φ.

Answer: $C = M(id)_{\mathcal{A}}^{st} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

b) by computing the characteristic polynomial as in a) we see that either $t = -2$ or $t = -3$. For $t = -3$ there exists a basis of $\mathbb{R}^3$ consisting of eigenvectors. It is enough to check that for $t = -2$.

$$V_{(-2)}: \begin{bmatrix} -2 & 1 & 2 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff x_2 = 0,\ x_1 = x_3,\ x_3 \in \mathbb{R}$$

$V_{(-2)} = \{(x_3, 0, x_3) \in \mathbb{R}^3 \mid x_3 \in \mathbb{R}\} = \operatorname{lin}((1,0,1))$

$$V_{(-3)}: \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff x_2 = 0, x_1 = 2x_3,\ x_3 \in \mathbb{R}$$

$V_{(-3)} = \{(2x_3, 0, x_3) \in \mathbb{R}^3 \mid x_3 \in \mathbb{R}\} = \operatorname{lin}((2,0,1))$
For $t = -2$ there is no basis of $\mathbb{R}^3$ consisting of eigenvectors of φ (too few linearly independent eigenvectors).
Answer: $t = -3$

Problem 4.
Let $\mathcal{A} = ((1,1,0),(0,0,1),(2,3,0))$ be an ordered basis of $\mathbb{R}^3$. The linear transformation $\psi\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is given by the matrix $M(\psi)_{\mathcal{A}}^{\mathcal{A}} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.

a) find $M(\psi)_{\mathcal{A}}^{st}$,
b) find formula of $\psi \circ \psi$.

Solution.

a) by definition of a matrix of a linear transformation

$$\begin{aligned}
\psi((1,1,0)) &= 1(1,1,0) - 1(0,0,1) + 0(2,3,0) = (1,1,-1),\\
\psi((0,0,1)) &= 1(1,1,0) + 0(0,0,1) - 1(2,3,0) = (-1,-2,0),\\
\psi((2,3,0)) &= 0(1,1,0) + 1(0,0,1) + 0(2,3,0) = (0,0,1).
\end{aligned}$$

Answer: $M(\psi)_{\mathcal{A}}^{st} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

b) From $a)$

$\psi((0,1,0)) = \psi((2,3,0)) - 2\psi((1,1,0)) = (0,0,1) - 2(1,1,-1) = (-2,-2,3).$

Therefore

$\psi((1,0,0)) = \psi((1,1,0)) - \psi((0,1,0)) = (1,1,-1) - (-2,-2,3) = (3,3,-4).$

Again, by definition

$$M(\psi)_{st}^{st} = \begin{bmatrix} 3 & -2 & -1 \\ 3 & -2 & -2 \\ -4 & 3 & 0 \end{bmatrix}.$$

$$M(\psi \circ \psi)_{st}^{st} = \begin{bmatrix} 3 & -2 & -1 \\ 3 & -2 & -2 \\ -4 & 3 & 0 \end{bmatrix} \begin{bmatrix} 3 & -2 & -1 \\ 3 & -2 & -2 \\ -4 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 7 & -5 & 1 \\ 11 & -8 & 1 \\ -3 & 2 & -2 \end{bmatrix}$$

Answer: $(\psi\circ\psi)(x_1,x_2,x_3) = (7x_1 - 5x_2 + x_3, 11x_1 - 8x_2 + x_3, -3x_1 + 2x_2 - 2x_3)$

Problem 5.
Let $V = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1 - x_2 + 2x_3 = 0\}$ be a subspace of $\mathbb{R}^3$.

a) find an orthonormal basis of $V^\perp$,
b) compute the orthogonal projection of $w = (3,0,0)$ onto V.

Solution.

a) treating coefficients of a system of linear equations as vectors spanning a subspace corresponds to passing from a vector space of solutions to its orthogonal completion. Therefore $V^\perp = \operatorname{lin}((1,-1,2))$.
Answer: $\mathcal{A} = (\frac{1}{\sqrt{6}}(1,-1,2))$ is the orthonormal basis of $V^\perp$

b) it is easier to project w onto $V^\perp$

$$P_{V^\perp}((3,0,0)) = \frac{(3,0,0)\cdot(1,-1,2)}{(1,-1,2)\cdot(1,-1,2)}(1,-1,2) = \frac{1}{2}(1,-1,2).$$

Since $P_V(w) + P_{V^\perp}(w) = w$ we have $P_V((3,0,0)) = (3,0,0) - \frac{1}{2}(1,-1,2)$.
Answer: $P_V((3,0,0)) = \frac{1}{2}(5,1,-2)$

Problem 6.
Let

$$A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

a) compute matrix AB,
b) compute $\det(B^4A^{-1} + B^5)$.

Solution.

a) compute $A = (A^{-1})^{-1}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 3 & 0 & 0 & 1 \end{array}\right] \overset{\text{easy row operations}}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

Therefore

$$AB = \begin{bmatrix} 3 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Answer: $AB = \begin{bmatrix} 3 & 0 & -4 \\ -1 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix}$

b)

$$\det(B^4A^{-1} + B^5) = \det(B^4(A^{-1} + B)) = (\det B)^4 \det \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix} \overset{w_3 - 4w_2}{=}$$

$$= (\det B)^4 \det \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ -2 & -9 & 0 \end{bmatrix} = 1^4 \cdot (-1)^{2+3}(-18+2) = 16.$$

Answer: $\det(B^4A^{-1} + B^5) = 16$

Problem 7.
Let $L \subset \mathbb{R}^3$ be an affine line given by the system of linear equations

$$\begin{cases} x_1 - x_3 = 2 \\ 2x_1 - x_2 = 3 \end{cases}$$

a) find a parametrization of L,
b) find an equation of the affine plane perpendicular to L passing through $(1,0,0)$.

Solution.

a) it is enough to solve the system of linear equations which is straightforward

$$\begin{cases} x_2 = 2x_1 - 3 \\ x_3 = x_1 - 2 \end{cases}, \quad x_1 \in \mathbb{R}$$

The general solution can be presented as $(x_1, 2x_1 - 3, x_1 - 2) = (0, -3, -2) + x_1(1, 2, 1),\ x_1 \in \mathbb{R}$.

Answer: parametrization of L: $(0, -3, -2) + t(1, 2, 1),\ t \in \mathbb{R}$.

b) since $\overrightarrow{L} = \operatorname{lin}(1, 2, 1)$ then $\overrightarrow{L}^{\perp}$ is equal to the set of solutions of the equation $x_1 + 2x_2 + x_3 = 0$.

We need to modify the constant term so the plane passes through $(1, 0, 0)$.

Answer: $x_1 + 2x_2 + x_3 = 1$

Problem 8.

Consider the following linear programming problem $-4x_1 - 3x_2 + 5x_3 - 2x_5 \to \min$ in the standard form with constraints

$$\begin{cases} x_1 + x_2 - x_3 + x_4 = 3 \\ 2x_1 + x_2 - 2x_3 + x_5 = 4 \end{cases} \quad \text{and } x_i \geqslant 0 \text{ for } i = 1, \dots, 5$$

a) which of the sets $\mathcal{B}_1 = \{1, 3\}, \mathcal{B}_2 = \{2, 3\}, \mathcal{B}_3 = \{4, 5\}$ are basic? Which basic sets are feasible?

b) solve the linear programming problem using simplex method.